

Definitions regarding (for $f: D \rightarrow \mathbb{R}$), $l \in \mathbb{R}$

$$\lim_{x \rightarrow x_0} f(x) = \begin{cases} l \\ +\infty \\ -\infty \end{cases} \quad (x_0, \text{cluster w.r.t. } D)$$

$$\lim_{x \rightarrow x_0^+} f(x) = \begin{cases} l \\ +\infty \\ -\infty \end{cases} \quad (\text{right-hand limit, } x_0 \text{ cluster w.r.t. } D \cap (x_0, +\infty))$$

$$\lim_{x \rightarrow x_0^-} f(x) = \begin{cases} l \\ +\infty \\ -\infty \end{cases} \quad (\text{left-hand } \dots)$$

$$\lim_{x \rightarrow +\infty} f(x) = \begin{cases} l \\ +\infty \\ -\infty \end{cases} \quad \left(\begin{array}{l} +\infty, \text{ "cluster" w.r.t. } D: \\ \forall a \in \mathbb{R} \exists x \in D \text{ s.t. } x \geq a \end{array} \right)$$

$$\lim_{x \rightarrow -\infty} f(x) = \begin{cases} l \\ +\infty \\ -\infty \end{cases} \quad \left(\forall a \in \mathbb{R} \exists x \in D \text{ s.t. } x \leq a \right)$$

Theorems: Uniqueness, but no "local boundedness result"
careful with computation rules.

Examples

$$1 \quad \lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty, \quad \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty, \quad \lim_{x \rightarrow 0} \frac{1}{x} \text{ not exist.}$$

The last assertion follows from the following theorem:

Theorem (on one-sided limits). Let $\lim_{x \rightarrow x_0^-} f(x) = l_1$ and

$$\lim_{x \rightarrow x_0^+} f(x) = l_2 \text{ where } l_1, l_2 \in [-\infty, \infty]. \text{ If } l_1 = l_2$$

then $\lim_{x \rightarrow x_0} f(x)$ exists and equals $l_1 = l_2$.

Conversely, if $\lim_{x \rightarrow x_0} f(x) = l$ exists in $[-\infty, \infty]$, then $\lim_{x \rightarrow x_0^-} f(x) = l = \lim_{x \rightarrow x_0^+} f(x)$.

$$1. \lim_{x \rightarrow 3} \frac{x+1}{x-2} = 4$$



$$\left| \frac{x+1}{x-2} - 4 \right| = \left| \frac{-3x+9}{x-2} \right| = \left| \frac{-3(x-3)}{x-2} \right| \leq \frac{3|x-3|}{|x-2|} \quad (x \neq 2)$$

(1) Why the following answer is wrong? Let $\varepsilon > 0$

Take $\delta = \frac{\varepsilon}{3} |x-2|$. Then if $|x-3| < \delta$ then

$$\left| \frac{x+1}{x-2} - 4 \right| \leq \frac{3|x-3|}{|x-2|} < \frac{3 \cdot \frac{\varepsilon}{3} |x-2|}{|x-2|} = \varepsilon$$

(what happens if x' is different from x and $|x'-2| < \delta$?)

(2) How to think for a solution. Let $\varepsilon > 0$ in attempting to choose a

good $\delta > 0$, one has to ensure that for $\forall x \in V_\delta(3) = (3-\delta, 3+\delta)$

is of (strictly) positive distance from 2 (so as

the factor $|x-2|$ (appeared in the denominator)

is bounded ^{below} by a positive constant $c > 0$, $\forall x \in V_\delta(3)$

$$|x-2| \geq c \quad \text{i.e.} \quad \frac{1}{|x-2|} \leq \frac{1}{c}$$

$$\forall x \in V_\delta(3)$$

$$\text{Now, with } \delta > 0, \quad \text{dist}(2, V_\delta(3)) = \begin{cases} 0 & \text{if } \delta \geq 1 \\ 1-\delta & \text{if } \delta < 1 \end{cases}$$

we are led to require that $\delta < 1$ (in addition to being positive). Having set this "preliminary condition"

one notes that, $\forall x \in V_\delta(3) = (3-\delta, 3+\delta)$, one has

$$|x-2| = x-2 > (3-\delta)-2 = 1-\delta$$

and so

$$\frac{3|x-3|}{|x-2|} \leq \frac{3|x-3|}{1-\delta} < \frac{3\delta}{1-\delta} \leq \epsilon$$

if $1-\delta \geq c (>0)$ and $\delta \leq \frac{c\epsilon}{3}$. Thus

one can take ^{p.g} $\delta = \min\left\{\frac{1}{2}, \frac{\epsilon}{6}\right\}$ or, for another choices

$$\delta = \min\left\{\frac{1}{3}, \frac{2\epsilon}{9}\right\}$$

$$\delta = \min\left\{\frac{2}{3}, \frac{\epsilon}{9}\right\} \text{ etc.}$$

(3) Formal Solution. Let $\epsilon > 0$. Let δ be defined by

$$\delta = \min\left\{\frac{1}{2}, \frac{\epsilon}{6}\right\} (>0)$$

Let $x \in V_\delta(3) = (3-\delta, 3+\delta)$. ~~It~~ suffices to show that

$$(*) \quad \left| \frac{x+1}{x-2} - 4 \right| < \epsilon$$

To do this, note that

$$3 - \frac{1}{2} < 3 - \delta < x < 3 + \delta \leq 3 + \frac{1}{2}$$

and so

$$\frac{1}{2} < x-2 < \frac{3}{2}$$

$$\text{Hence } \left| \frac{x+1}{x-2} - 4 \right| = \left| \frac{x+1-4(x-2)}{x-2} \right| = \left| \frac{-3x+9}{x-2} \right| = \frac{3|x-3|}{|x-2|}$$

$$< \frac{3|x-3|}{\frac{1}{2}} = 6|x-3| < 6\delta \leq 6 \cdot \frac{\epsilon}{6} = \epsilon,$$

verifying (*).

$$2. \lim_{x \rightarrow 3} \frac{x^3+1}{x-2} = 28$$

Let $\varepsilon > 0$. Let $\delta > 0$ be defined by

$$\delta := \min \left\{ \frac{1}{2}, \frac{\varepsilon}{100} \right\}$$

Let x be s.t. $|x-3| < \delta$. It suffices to show that:

$$(*) \quad \left| \frac{x^3+1}{x-2} - 28 \right| < \varepsilon.$$

Note that

$$|x-3| < \delta \leq \frac{1}{2} \quad \text{so} \quad \frac{5}{2} < x < 3\frac{1}{2}$$

and hence

$$\frac{1}{2} \leq |x^2-2| \quad \& \quad |x| < 4$$

Consequently

$$\left| \frac{x^3+1}{x-2} - 28 \right| = \frac{|x^3+1-28(x-2)|}{|x-2|} = \frac{|x-3| |x^2+3x-19|}{|x-2|}$$

$$\leq \frac{|x-3| \cdot (|x|^2 + 3|x| + 19)}{\frac{1}{2}} \leq 100|x-3| \leq 100\delta \leq \varepsilon,$$

verifying (*).